Support of dS/CFT correspondence from perturbations of three dimensional spacetime

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Abstract

We discuss the relation between bulk de Sitter three-dimensional spacetime and the corresponding conformal field theory at the boundary, in the framework of the exact quasinormal mode spectrum. We show that the quasinormal mode spectrum corresponds exactly to the spectrum of thermal excitations of Conformal Field Theory at the past boundary I^-, together with the spectrum of a Conformal Field Theory at the future boundary I^+.

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The study of quasinormal modes has been an intriguing research activity for the last 30 years [1], leading to important contributions to the understanding of black holes [2-11]. Recently the case of Anti-de Sitter (AdS) space has been particularly focused due to its proposed relation to the conformal field theory (CFT) [12]. Qualitative correspondences beween quasinormal modes in AdS spaces and the decay of perturbations in the due CFT have been obtained [5-11]. More encouraging, in the three-dimensional (3D) BTZ black hole model [13], a precise quantitative agreement between the quasinormal frequencies and the location of poles of the retarded correlation function of the corresponding perturbations in the dual CFT has been presented [11]. This gives a further evidence of the correspondence between gravity in AdS spacetime and quantum field theory at the boundary.

There has been an increasing interest in gravity on de Sitter (dS) spacetimes in view of recent observational support for a positive cosmological constant. A holographic duality relating quantum gravity on D-dimensional dS space to CFT on (D-1)-sphere has been proposed [14]. It is of interest to extend the study in [11] to dS space by displaying the exact solution of the quasinormal mode problem in the dS bulk space and exploring its relation to the CFT theory at the boundary. This could serve as a quantitative test of the dS/CFT correspondence. This is the motivation of the present paper. We will concentrate on nontrivial 3D dS spacetimes. The mathematical simplicity in these models renders all computations analytical. As shown in [15] [16] 3D gravity is directly related to the two-dimensional (2D) WZW theory at the border [17], where the whole theoretical apparatus of CFT can be fully used to obtain exact results. In such a case, Brown and Henneauax [18] already obtained one relation between 3D AdS gravity and conformal algebra with a (classical version of the) central charge, recently developed into the quantum theory by Maldacena [12].

The metric of the 3D rotating dS spacetime is given by

$$ds^2 = -(M - \frac{r^2}{l^2} + \frac{J^2}{4r^2})dt^2 + (M - \frac{r^2}{l^2} + \frac{J^2}{4r^2})^{-1}dr^2 + r^2(d\varphi - \frac{J}{2r^2}dt)^2, \tag{1}$$

where J is associated to the angular momentum. The horizon of such spacetime can be obtained from

$$M - \frac{r^2}{l^2} + \frac{J^2}{4r^2} = 0. \tag{2}$$

The solution is given in terms of r_+ and $-ir_-$, where r_+ corresponds to the cosmological horizon and $-ir_-$ here being imaginary, has no physical interpretation in terms of a horizon. Using r_+ and r_-, the mass and angular momentum of spacetime can be expressed as

$$M = \frac{r_+^2 - r_-^2}{l}, \qquad J = \frac{-2r_+r_-}{l} \tag{3}$$

Scalar perturbations of this spacetime are described by the wave equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - \mu^2\Phi = 0, \tag{4}$$

where μ is a mass of the field. Adopting the separation

$$\Phi(t, r, \varphi) = R(r)\ e^{-i\omega t}\ e^{im\varphi}, \tag{5}$$

the radial part of the wave equation can be written as

$$\frac{1}{g_{rr}}\frac{d}{rdr}(\frac{r^2}{g_{rr}}\frac{dR}{rdr}) + [\omega^2 - \frac{1}{r^2}m^2(M - \frac{r^2}{l^2}) - \frac{J}{r^2}m\omega]R = \frac{1}{g_{rr}}\mu^2 R, \tag{6}$$

where $g_{rr} = (M - r^2/l^2 + J^2/(4r^2))^{-1}$. Employing (3) and defining $z = \dfrac{r^2 - r_+^2}{r^2 - (-ir_-)^2}$, the radial wave equation can be simplified into

$$(1-z)\frac{d}{dz}(z\frac{dR}{dz}) + \left[\frac{1}{z}\left(\frac{\omega l^2 r_+ + mlr_-}{2(r_+^2 + r_-^2)}\right)^2 - \left(\frac{-\omega l^2 ir_- + imlr_+}{2(r_+^2 + r_-^2)}\right)^2 + \frac{1}{4(1-z)}\mu^2 l^2\right] R = 0. \tag{7}$$

We now set the Ansatz

$$R(z) = z^\alpha(1-z)^\beta F(z), \tag{8}$$

and Eq(7) can be transformed into

$$\begin{aligned} z(1-z)\frac{d^2F}{dz^2} \quad &+ \quad [1 + 2\alpha - (1 + 2\alpha + 2\beta)z]\,\frac{dF}{dz} \\ &+ \quad [(\beta(\beta-1) + \frac{\mu^2 l^2}{4})\frac{1}{1-z} + \frac{1}{z}\left[\left(\frac{\omega l^2 r_+ + mlr_-}{2(r_+^2 + r_-^2)}\right)^2 + \alpha^2\right] \\ &- \quad \left[\left(\frac{-i\omega l^2 r_- + imlr_+}{2(r_+^2 + r_-^2)}\right)^2 + \alpha^2 + (1 + 2\alpha)\beta + \beta(\beta - 1)\right]]F = 0. \end{aligned} \tag{9}$$

Comparing with the standard hypergeometric equation

$$z(1-z)\frac{d^2F}{dz^2} + [c-(1+a+b)z]\frac{dF}{dz} - abF = 0, \tag{10}$$

we have

$$\begin{aligned}
c &= 1+2\alpha \\
a+b &= 2\alpha+2\beta \\
\alpha^2 + \left(\frac{\omega l^2 r_+ + mlr_-}{2(r_+^2+r_-^2)}\right)^2 &= 0 \\
\beta(\beta-1) + \frac{\mu^2 l^2}{4} &= 0 \\
ab &= \left(\frac{-\omega l^2 ir_- + imlr_+}{2(r_+^2+r_-^2)}\right)^2 + (\alpha+\beta)^2.
\end{aligned} \tag{11}$$

Without loss of generality, we can take

$$\alpha = -i\left(\frac{\omega l^2 r_+ + mlr_-}{2(r_+^2+r_-^2)}\right), \qquad \beta = \frac{1}{2}\left(1-\sqrt{1-\mu^2 l^2}\right), \tag{12}$$

which leads to

$$\begin{aligned}
a &= -\frac{i}{2}\left(\frac{\omega l^2 + iml}{r_+ + ir_-} + i(1-\sqrt{1-\mu^2 l^2})\right), \\
b &= -\frac{i}{2}\left(\frac{\omega l^2 - iml}{r_+ - ir_-} + i(1-\sqrt{1-\mu^2 l^2})\right), \\
c &= 1 - i\left(\frac{\omega l^2 r_+ + mlr_-}{r_+^2+r_-^2}\right),
\end{aligned} \tag{13}$$

and the solution of (7) reads

$$R(z) = z^\alpha (1-z)^\beta \, {}_2F_1(a,b,c,z) \tag{14}$$

Using basic properties of the hypergeometric equation we write the result as

$$\begin{aligned}
R(z) &= z^\alpha(1-z)^\beta(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \, {}_2F_1(c-a,c-b,c-a-b+1,1-z) \\
&\quad + z^\alpha(1-z)^\beta\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \, {}_2F_1(a,b,a+b-c+1,1-z)
\end{aligned} \tag{15}$$

The first term vanishes at $z=1$, while the second vanishes provided that

$$c-a=-n, \qquad or \qquad c-b=-n, \tag{16}$$

where $n = 0, 1, 2, ...$ Employing Eqs(13), it is easy to see that the quasinormal frequencies are

$$\begin{aligned}
\omega_R &= i\frac{m}{l} - 2i\left(\frac{r_+ - ir_-}{l^2}\right)(n + \frac{1}{2} + \frac{1}{2}\sqrt{1 - \mu^2 l^2}) \\
\omega_L &= -i\frac{m}{l} - 2i\left(\frac{r_+ + ir_-}{l^2}\right)(n + \frac{1}{2} + \frac{1}{2}\sqrt{1 - \mu^2 l^2}).
\end{aligned} \tag{17}$$

Taking other values of α and β satisfying (11), we have also the frequencies

$$\begin{aligned}
\omega_R &= i\frac{m}{l} + 2i\left(\frac{r_+ - ir_-}{l^2}\right)(n + \frac{1}{2} + \frac{1}{2}\sqrt{1 - \mu^2 l^2}) \\
\omega_L &= -i\frac{m}{l} + 2i\left(\frac{r_+ + ir_-}{l^2}\right)(n + \frac{1}{2} + \frac{1}{2}\sqrt{1 - \mu^2 l^2}),
\end{aligned} \tag{18}$$

$$\begin{aligned}
\omega_R &= i\frac{m}{l} - 2i\left(\frac{r_+ - ir_-}{l^2}\right)(n + \frac{1}{2} - \frac{1}{2}\sqrt{1 - \mu^2 l^2}) \\
\omega_L &= -i\frac{m}{l} - 2i\left(\frac{r_+ + ir_-}{l^2}\right)(n + \frac{1}{2} - \frac{1}{2}\sqrt{1 - \mu^2 l^2}),
\end{aligned} \tag{19}$$

$$\begin{aligned}
\omega_R &= i\frac{m}{l} + 2i\left(\frac{r_+ - ir_-}{l^2}\right)(n + \frac{1}{2} - \frac{1}{2}\sqrt{1 - \mu^2 l^2}) \\
\omega_L &= -i\frac{m}{l} + 2i\left(\frac{r_+ + ir_-}{l^2}\right)(n + \frac{1}{2} - \frac{1}{2}\sqrt{1 - \mu^2 l^2}).
\end{aligned} \tag{20}$$

Now let us investigate quasinormal modes from the CFT side. It is believed that for a thermodynamical system the relaxation process of a small perturbation is completely determined by the poles, in the momentum representation, of the retarded correlation function of the perturbation. The correlation functions in dS spacetimes have been studied in [19] [14].

By describing the coordinates in $SO(3,1)$ [15] such that

$$X_1^2 + X_2^2 + X_3^2 - T^2 = l^2. \tag{21}$$

The metric (1) can be reobtained by the change of variables

$$\begin{aligned}
X_1 &= l\sqrt{\chi}\sin(\frac{r_+}{l}\varphi - \frac{r_-}{l^2}t) && (22) \\
X_2 &= -l\sqrt{1-\chi}\cosh(\frac{r_+}{l^2}t + \frac{r_-}{l}\varphi) \\
X_3 &= l\sqrt{\chi}\cos(\frac{r_+}{l}\varphi - \frac{r_-}{l^2}t) && (23) \\
T &= -l\sqrt{1-\chi}\sinh(\frac{r_+}{l^2}t + \frac{r_-}{l}\varphi)
\end{aligned}$$

where $\chi = \dfrac{r^2 + r_-^2}{r_+^2 + r_-^2}$.

The invariant distance between two points defined by x and x' reads [14][19]

$$d = l \arccos P, \tag{24}$$

where $P = X^A \eta_{AB} X'^B$. In the limit $r, r' \to \infty$,

$$P \approx 2 \sinh \frac{(ir_+ + r_-)(l\Delta\varphi - i\Delta t)}{2l^2} \sinh \frac{(ir_+ - r_-)(l\Delta\varphi + i\Delta t)}{2l^2} \tag{25}$$

This means that we can find the Hadamard Green's function as defined by [19] in terms of P. Such a Green's function is defined as $G(u, u') =< 0| \{\phi(u), \phi(u')\} |0>$ with $(\nabla_x^2 - \mu^2)G = 0$. It is possible to obtain the solution

$$G \sim F(h_+, h_-, 3/2, (1 + P)/2) \tag{26}$$

in the limit $r, r' \to \infty$, where $h_\pm = 1 \pm \sqrt{1 - \mu^2 l^2}$.

Following [14] [19], we choose boundary conditions for the fields such that

$$\lim_{r\to\infty} \phi(r, t, \varphi) \to r^{-h_-} \phi_-(t, \varphi) \tag{27}$$

Then, for large r, r', an expression for the two point function of a given operator O coupling to ϕ has the form

$$\begin{aligned} \lim_{r\to\infty} \quad & \int dtd\varphi dt' d\varphi' \frac{(rr')^2}{l^2} \phi \overleftrightarrow{\partial}_{r*} G \overleftrightarrow{\partial}_{r*} \phi \qquad (28) \\ = \quad & \int dtd\varphi dt' d\varphi' \phi \frac{1}{[2 \sinh \frac{(ir_+ + r_-)(l\Delta\varphi - i\Delta t)}{2l^2} \sinh \frac{(ir_+ - r_-)(l\Delta\varphi + i\Delta t)}{2l^2}]^{h_+}} \phi \end{aligned}$$

where $r*$ in (28) is the tortoise coordinate.

For quasinormal modes, we have

$$\begin{aligned} & \int dtd\varphi dt' d\varphi' \frac{exp(-im'\varphi' - i\omega' t' + im\varphi + i\omega t)}{[2 \sinh \frac{(ir_+ + r_-)(l\Delta\varphi - i\Delta t)}{2l^2} \sinh \frac{(ir_+ - r_-)(l\Delta\varphi + i\Delta t)}{2l^2}]^{h_+}} \\ \approx \quad & \delta_{mm'} \delta(\omega - \omega') \Gamma(h_+/2 + \frac{im/2l + \omega/2}{2\pi T}) \Gamma(h_+/2 - \frac{im/2l + \omega/2}{2\pi T}) \\ \times \quad & \Gamma(h_+/2 + \frac{im/2l - \omega/2}{2\pi \bar{T}}) \Gamma(h_+/2 - \frac{im/2l - \omega/2}{2\pi \bar{T}}) \end{aligned} \tag{29}$$

where we changed variables to $v = l\varphi + it, \bar{v} = l\varphi - it$, and $T = \dfrac{ir_+ - r_-}{2\pi l^2}, \bar{T} = \dfrac{ir_+ + r_-}{2\pi l^2}$. The poles of such a correlator are

$$\begin{aligned} \omega_L &= -\frac{im}{l} \pm 2\frac{ir_+ - r_-}{l^2}(n + h_+/2), \\ \omega_R &= \frac{im}{l} \pm 2\frac{ir_+ + r_-}{l^2}(n + h_+/2), \end{aligned} \tag{30}$$

corresponding to the quasinormal modes (17,18) obtained before.

As stressed by Strominger [14], the boundary condition (27) is not mandatory, and we may choose $\phi(r,t,\varphi) \sim r^{-h_+}\phi(t,\varphi)$ at infinity. This leads to correlations similar to those found up to now with the change $h_+ \to h_-$, and we complete the set of quasinormal frequencies in (19,20).

The dS Green functions display two symmetries, which in terms of the variable v can be described as

$$\delta v = \frac{2\pi l^2}{r_+^2 - r_-^2}\left[n(r_+ + ir_-) + n'(r_- + ir_+)\right] \tag{31}$$

for n, n' arbitrary integers. We thus have a two dimensional lattice, and a fundamental region defines the full theory.

The quasinormal eigenfunctions thus correspond to excitation of the corresponding CFT, being exactly those that appear in the spectrum of the two point functions of CFT operators in dS background for large values of r, that is at the boundary.

There are some complications compared to the AdS case. There, it has well defined temperature for the theory at the border, given by $T_\pm^{AdS} = (r_+ \pm r_-)/(2\pi l^2)$, where r_+ and r_- are the (real) values for the horizons. However, in dS case only the cosmological horizon r_+ exists, while r_- being imaginary. Thus the attempt to define temperature leads us to compute values for it $T_\pm^{dS} = (ir_+ \pm r_-)/(2\pi l^2)$.

It is worth noticing that since we expect instability for the black hole formation in dS space. The Choptuik parameter γ cannot be defined. While in 3D AdS BTZ case it takes the value 1/2 [20]. We cannot apply the same method here. In particular, the relation found in [6] between the imaginary part of the frequency, γ and r_+ in the form $\omega_{im} \sim r_+/\gamma$ cannot hold here, since now the dS radius l enters nontrivally, that is $\omega_{im} \approx m/l + r_+(n + h_+/2)/l^2$. We are thus in a phase where a black hole can simply not be formed. An independent proof of this statement would be welcome.

Finally, comparing with Strominger's results, we here support the picture of CFT at a boundary of de Sitter space, finding nevertheless excitations at both boundaries, that is, half of them in the past boundary and the other half in the future boundary. Concluding, we found that bulk dS space quasinormal modes can be described in terms of a thermal two-dimensional gas with a complete parameters describing instabilities of the bulk matter. The results obtained here provide a quantitative support of the dS/CFT correspondence.

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